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Conservative and Entropy Decaying Numerical Scheme for the Isotropic Fokker–Planck–Landau Equation

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Homogeneous Fokker–Planck–Landau equation denoted by FPLE is studied for Coulombian and isotropic distribution function, i.e. when the distribution function depends only on time and on the modulus of the velocity. We derive a new conservative and entropy decaying semi-discretized FPLE for which we prove the existence of global in time, positive. For the time-discretized equation, we give upper bound for the time step which guarantes positivity and entropy decay of the numerical solution. © 1998 Academic Press

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1. INTRODUCTION

The FPLE is commonly used in plasma physics when studying kinetical effects between charged particles under Coulomb interaction. The homogeneous isotropic FPLE descibes thermalization processes of the plasma in isotropic situations for the velocity variable and independent of the space variable. Another interest of the FPLE is to produce precise solutions in order to study numerical schemes in the 3D velocity space [4, 5, 8, 16–18] or in the 2D axisymmetric case [15]. Indeed, no explicit solutions are known for the Coulomb potential case $\gamma = -3$, defined in Section 2, contrary to the Maxwellian case $\gamma = 0$ [13]. There are also applications in the astrophysics field, where the FPLE is used for star cluster modelling [6, 7].

Existence results for the continuous FPLE can be found in [9, 10, 1]. These results can certainly be extended for the isotropic equation considered here.

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A conservative and entropy scheme for the (spherical and homogeneous) FPLE was first proposed by Berezin, Khudick, and Pekker [2]. They give an upper bound for the time step to ensure the decay of entropy without a complete proof of their assertion. Entropy decay is physically relevant and seems to prevent oscillations (as shown in the sequel on numerical examples and proved for the linear case in [4]). At the continuous level and for obvious physical reason, the solution remains positive at any time. Thus, the discretization must preserve this property and this does not appear clearly in [2]. See [4] for an example of a conservative discretization which does not preserve positivity for all positive initial data. In this work, we prove the positivity of the solution for the semi-discretized and time-discretized solution for arbitrarily large time.

The aim of this paper is to propose a new conservative and entropy decaying scheme for FPLE for which, first, we prove the existence of a unique and global in time solution for the semi-discretized problem and, second, for the time-discretized equation we exhibit an upper bound on the time step to ensure the positivity and the decay of the entropy. Moreover, we show that the cost of the numerical evaluation of this operator is proportional to the number of discretization points despite its quadratic structure. Let us also mention that this scheme can be considered on an arbitrary mesh, contrary to the discretization considered in [4, 5, 14]. This last property permits us to refine the mesh size for small velocity and, thus, to obtain more accurate solutions. However, some questions remain open like the long-time behaviour of the semi-discretized or time-discretized solution, although it is expected that the distribution function converges to the discretized Maxwellian.

2. THE HOMOGENEOUS AND ISOTROPIC FPLE

We denote by $F(\mathbf{v}, t)$ the distribution function solution of the scaled integro-differential equation

$$\frac{\partial F}{\partial t} = Q(F, F) = \nabla_{\mathbf{v}}.\left(\int_{\mathbb{R}^3} \Phi(\mathbf{v} - \mathbf{v}_*)\big((\boldsymbol{\nabla}_{\mathbf{v}} F)F_* - \big(\boldsymbol{\nabla}_{\mathbf{v}_*} F\big)F\big)\, d\mathbf{v}_*\right), \tag{2.1}$$

where $Q(F, F)$ is the Fokker–Planck collision operator written in the so called Landau form with the standard notations (for example $F_* = F(\mathbf{v}_*, t)$) and $\Phi(\mathbf{v})$ is the following 3×3 matrix:

$$\Phi(\mathbf{v}) = |\mathbf{v}|^{\gamma+2} S(\mathbf{v}), \quad S(\mathbf{v}) = I_3 - \frac{\mathbf{v} \otimes \mathbf{v}}{|\mathbf{v}|^2}. \tag{2.2}$$

$S(\mathbf{v})$ is the orthogonal projector onto the plane orthogonal to $\mathbf{v}$. γ is a real parameter which leads to the usual classification in hard potentials ($\gamma > 0$), maxwellian molecules ($\gamma = 0$) or soft potentials ($\gamma < 0$). This latter case involves the Coulomb case (i.e., $\gamma = -3$) which is of primary importance for plasma applications. The well-known physical properties of (2.1) are similar to that of the Boltzmann operator such as the decay of the entropy, the conservation of mass, momentum, and energy, and the characterization of the equilibrium states by Maxwellians. We refer to [8, 16] for a detailed presentation of this equation.

It can be easily check that isotropic initial data leads to an isotropic solution for the classical nonlinear FPLE. In other words, if the distribution function $F(\mathbf{v}, t)$ depends only of the modulus of the velocity $v = \|\mathbf{v}\|$ at time $t = 0$, then this holds for any arbitrary time t; i.e., there exists a function f such that $F(\mathbf{v}, t) = f(v, t)$ (see [2, 17, 18]). In the Coulomb case, such isotropic distribution function $f(\varepsilon, t)$, where $\varepsilon = v^2$ is the energy variable, satisfies a dimensionless equation of the form:

$$\frac{\partial f}{\partial t} = \frac{1}{\sqrt{\varepsilon}} \frac{\partial}{\partial \varepsilon} \int_0^{\infty} f(\varepsilon) f(\varepsilon') \left(\frac{\partial}{\partial \varepsilon} \ln f(\varepsilon) - \frac{\partial}{\partial \varepsilon} \ln f(\varepsilon') \right) k(\varepsilon, \varepsilon') \, d\varepsilon'. \tag{2.3}$$

For numerical simulations, we reduce the integration domain in FPLE to a bounded domain in the variable ε as in [2] :

$$\frac{\partial f}{\partial t} = \frac{1}{\sqrt{\varepsilon}} \frac{\partial}{\partial \varepsilon} \int_0^{\varepsilon_0} f(\varepsilon) f(\varepsilon') \left(\frac{\partial}{\partial \varepsilon} \ln f(\varepsilon) - \frac{\partial}{\partial \varepsilon} \ln f(\varepsilon') \right) k(\varepsilon, \varepsilon') \, d\varepsilon', \tag{2.4}$$

where $k(\varepsilon, \varepsilon') = \inf(\varepsilon^{3/2}, (\varepsilon')^{3/2})$ and ε_0 is choosen such that the distribution function is near zero outside the ball of radius ε_0. Physically, ε_0 is choosen larger than the typical scaled energy. We refer to [2] for a physical justification of this scaling. This operator can be equivalently written in the following weak form: for any sufficiently smooth and decaying test function $\phi(\varepsilon)$,

$$\begin{aligned} \int_0^{\varepsilon_0} \frac{\partial f}{\partial t} \phi \sqrt{\varepsilon} \, d\varepsilon = -\frac{1}{2} \int_0^{\varepsilon_0} \int_0^{\varepsilon_0} f(\varepsilon) f(\varepsilon') \left(\frac{\partial \phi(\varepsilon)}{\partial \varepsilon} - \frac{\partial \phi(\varepsilon')}{\partial \varepsilon} \right) \\ \times \left(\frac{\partial \ln f(\varepsilon)}{\partial \varepsilon} - \frac{\partial \ln f(\varepsilon')}{\partial \varepsilon} \right) k(\varepsilon, \varepsilon') \, d\varepsilon' \, d\varepsilon. \end{aligned} \tag{2.5}$$

This operator satisfies the conservation of mass (resp. energy) by choosing $\phi = 1$ (resp. $\phi = \varepsilon$ in (2.5))

$$\rho = \int_0^{\varepsilon_0} f(\varepsilon) \sqrt{\varepsilon} \, d\varepsilon, \tag{2.6}$$

$$\rho E = \int_0^{\varepsilon_0} f(\varepsilon) \varepsilon^{3/2} \, d\varepsilon. \tag{2.7}$$

The entropy defined by

$$H = \int_0^{\varepsilon_0} f(\varepsilon) \ln(f(\varepsilon)) \sqrt{\varepsilon} \, d\varepsilon \tag{2.8}$$

decays with time (by letting $\phi = \ln(f)$ in the weak formulation of FPLE) and satisfies the classical H theorem

$$\partial_t H = 0 \Leftrightarrow f = \exp(-A\varepsilon + B).$$

3. THE SEMI-DISCRETIZED FPLE

Let us introduce the discretization $f_i = f(\varepsilon_i)$, where $(\varepsilon_i)_{i=1 \cdots N}$ is an increasing sequence such that $\varepsilon_1 = 0$, $\varepsilon_N = \varepsilon_0$, and $(\Delta \varepsilon_i = (\varepsilon_{i+1} - \varepsilon_i))_{i=1 \cdots N-1}$, is also increasing. The ε-derivative are approximated according to the simplest choice of finite difference operator namely, we define for any discretized function $(\phi_i)_{i=1 \cdots N}$

$$\mathrm{D}\phi_i = \frac{(\phi_{i+1} - \phi_i)}{\Delta \varepsilon_i}, \quad i = 1 \cdots N - 1.$$

Let us introduce some notations. We define $\varepsilon_{i+1/2} = (\varepsilon_{i+1} + \varepsilon_i)/2$ and $v_{i+1/2}$ as the mean value of the velocity on $[\varepsilon_i, \varepsilon_{i+1}]$, i.e.

$$v_{i+1/2} = \frac{1}{\Delta\varepsilon_i} \int_{\varepsilon_i}^{\varepsilon_{i+1}} \sqrt{\varepsilon}\, d\varepsilon = \frac{2}{3\Delta\varepsilon_i} \big(\varepsilon_{i+1}^{3/2} - \varepsilon_i^{3/2}\big).$$

Let us consider first the discretization of the expression $\int_0^{\varepsilon_0} \phi\sqrt{\varepsilon}\, d\varepsilon$ for any function ϕ. By writing

$$\int_0^{\varepsilon_0} \phi\sqrt{\varepsilon}\, d\varepsilon = \sum_{i=1}^{N-1} \int_{\varepsilon_i}^{\varepsilon_{i+1}} \phi\sqrt{\varepsilon}\, d\varepsilon$$

and using the trapezoidal quadrature formula with respect to the measure $\sqrt{\varepsilon}\, d\varepsilon$, we approximate it by

$$\sum_{i=1}^{N-1} \frac{1}{2} (\phi_i + \phi_{i+1}) v_{i+1/2} \Delta\varepsilon_i .$$

By factorizing the terms ϕ_i in the above expression, we obtain

$$\frac{1}{2}\phi_1 v_{3/2} \Delta\varepsilon_1 + \frac{1}{2} \sum_{i=2}^{N-1} \phi_i (v_{i+1/2}\Delta\varepsilon_i + v_{i-1/2}\Delta\varepsilon_{i-1}) + \frac{1}{2}\phi_N v_{N-1/2} \Delta\varepsilon_N \overset{def}{=} \sum_{i=1}^{N} c_i \phi_i, \quad (3.1)$$

where c_i are such that $c_1 = v_{3/2}\Delta\varepsilon_1/2 = \frac{1}{3}\varepsilon_2^{3/2}$,

$$c_i = \frac{1}{2}(v_{i+1/2}\Delta\varepsilon_i + v_{i-1/2}\Delta\varepsilon_{i-1}) = \frac{1}{3}\big(\varepsilon_{i+1}^{3/2} - \varepsilon_{i-1}^{3/2}\big),$$

for $i = 2 \cdots N - 1$ and $c_N = v_{N-1/2}\Delta\varepsilon_{N-1}/2 = \frac{1}{3}(\varepsilon_N^{3/2} - \varepsilon_{N-1}^{3/2})$. Once applied to the left-hand side of (2.5) with $(\partial f/\partial t)\phi$, we obtain the discretization of $\int_0^{\varepsilon_0} (\partial f/\partial t)\phi\sqrt{\varepsilon}\, d\varepsilon$ as $\sum_{i=1}^{N} c_i (\partial f_i/\partial t)\phi_i$. We now turn to the discretization of the right-hand side of (2.5),

$$\begin{aligned}(\text{r.h.s.}) = -\frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \int_{\varepsilon_i}^{\varepsilon_{i+1}} \int_{\varepsilon_j}^{\varepsilon_{j+1}} f(\varepsilon) f(\varepsilon') \left(\frac{\partial}{\partial\varepsilon}\phi(\varepsilon) - \frac{\partial}{\partial\varepsilon}\phi(\varepsilon') \right) \\ \times \left(\frac{\partial}{\partial\varepsilon} \ln f(\varepsilon) - \frac{\partial}{\partial\varepsilon} \ln f(\varepsilon') \right) k(\varepsilon, \varepsilon')\, d\varepsilon'\, d\varepsilon. \quad (3.2)\end{aligned}$$

Using for each integrals of (3.2) a midpoint quadrature formula, we approximate (3.2) by

$$-\frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} g_i g_j k_{i,j} \Delta\varepsilon_i \Delta\varepsilon_j (\mathrm{D}\phi_i - \mathrm{D}\phi_j)(\mathrm{D}(\ln f)_i - \mathrm{D}(\ln f)_j) \quad (3.3)$$

with

$$k_{i,j} = k\big(\varepsilon_{i+1/2}^{3/2}, \varepsilon_{j+1/2}^{3/2}\big),$$

and the terms g_i stand for a second-order approximation of the distribution function at the center of the interval $[\varepsilon_i, \varepsilon_{i+1}]$. In the paper of Berezin *et al.* [2], the terms g_i are taken as an arithmetic mean of f_i and f_{i+1}. This yields a discrete model for which it cannot be proved that the distribution function remains positive as it must be. We take a second-order approximation as the harmonic mean; that is,

$$g_i \stackrel{def}{=} \frac{2}{1/f_i + 1/f_{i+1}} = \frac{2 f_i f_{i+1}}{f_i + f_{i+1}}. \tag{3.4}$$

Such an approximation has been already used by the authors (see [4]) for the linear and 3D nonlinear cases of the Fokker–Planck–Landau equation and the resulting discretized models for which the existence of a global positive solution is proved. Note that $\mathrm{D}\phi_i$ is also a second-order approximation of the derivative in the center of the cell $[\varepsilon_i, \varepsilon_{i+1}]$. We shall denote by $D_{i,j}$ the terms $(\mathrm{D}(\ln f)_i - \mathrm{D}(\ln f)_j)$ for simplifying the notations. Hence, the weak formulation of the semi-discretized model reads

$$\sum_{i=1}^{N} c_i \frac{\partial f_i}{\partial t} \phi_i = -\frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} g_i g_j k_{i,j} \Delta\varepsilon_i \Delta\varepsilon_j (\mathrm{D}\phi_i - \mathrm{D}\phi_j) D_{i,j}. \tag{3.5}$$

By factorizing the terms ϕ_i in the right-hand side of (3.5), we get

$$\text{(r.h.s.)} = \sum_{i=2}^{N-1} \phi_i (p_i - p_{i-1}) + \phi_1 p_1 - \phi_N p_{N-1}$$

for all $i = 1 \cdots N-1$:

$$p_i \stackrel{\text{def}}{=} \sum_{j=1}^{N-1} g_i g_j k_{i,j} D_{i,j} \Delta\varepsilon_j. \tag{3.6}$$

Finally, by identifying the terms involving ϕ_i in (3.5), we obtain the system of ordinary differential equations (which is of the same form as in the 3D case presented in [4, 5]),

$$\frac{\partial f_i}{\partial t} = \mathrm{FP}_i^s, \quad i = 1 \cdots N, \tag{3.7}$$

with $\mathrm{FP}_1^s = p_1/c_1$, $\mathrm{FP}_i^s = (p_i - p_{i-1})/c_i$ for $i = 2 \cdots N-1$, and $\mathrm{FP}_N^s = -p_{N-1}/c_{N-1}$. The conservation laws imply that the discretized analogous of mass (2.6) and energy (2.7) defined as

$$\rho = \sum_{j=1}^{N} c_j f_j \quad \text{(mass)}, \quad \rho E = \sum_{j=1}^{N} c_j f_j \varepsilon_j \quad \text{(energy)},$$

are conserved through the evolution of the system. These conservation properties can be easily checked by taking $\phi_i = 1$ and $\phi_i = \varepsilon_i$ in (3.5). Moreover, the entropy decays using the discretized definition of the entropy (in the spherical case)

$$H = H(f_i) \stackrel{\text{def}}{=} \sum_{j=1}^{N} c_j f_j \ln(f_j). \tag{3.8}$$

The verification is straightforward using the weak discretized formulation (3.5) with test function $\phi_i = \ln(f_i)$. Note that in the present case, the conservations and entropy decay hold whatever the discretization grid is uniform or not, which is not the case in the 3D case [4, 5].

The existence of a positive global in time solution for this system follows exactly the same line as the one of the full 3D system [4].

THEOREM 3.1. *The Cauchy problem for the differential equation* (3.7) *with a strictly positive initial data admits a unique, positive and entropy solution for any time.*

Proof. The existence and unicity of the solution for small time is obtained using classical Cauchy Lipschitz theorem. Indeed, there is no singularity in this system in the logarithmic terms, using $f_i^0 > 0$. Thus, the existence of a solution global in time holds, provided that the solution cannot vanish in finite time at some points. We follow exactly the same lines as for the full 3D system [4]. Using mass conservation, showing that the f_i's cannot vanish in finite time is equivalent to checking that the function

$$K = \sup_{i=1}^{N-1}\left(\left|\frac{f_i}{f_{i+1}}\right|, \left|\frac{f_{i+1}}{f_i}\right|\right) \tag{3.9}$$

remains bounded in finite time. This function is convenient since these ratios actually appear in the $\mathrm{D}(\ln f)_i$ terms. We have the following estimates (which are no more true with the arithmetic average instead of (3.4)):

$$0 \le g_i \le 2 f_{i+1} \quad \text{or} \quad 2 f_i \quad \forall i = 1 \cdots N-1. \tag{3.10}$$

Using (3.9) and the mass conservation, we have the estimate for the terms p_i,

$$|p_i| \le C g_i \ln(K), \tag{3.11}$$

where C is a generic constant throughout the rest of the proof, depending on the number of grid points N, domain size ε_0, the grid ε_i, and the initial data $(f_i^0)_{i\in I}$. Indeed, we have the following upper bounds:

$$|D_{i,j}| \le 2 \sup_{i=1\cdots N-1} |\mathrm{D}(\ln f)_i| \le 2\ln(K), \quad i = 1\cdots N-1,$$

$$\sum_{j=1}^{N-1} g_j k_{i,j} \Delta\varepsilon_j \le \sum_{j=1}^{N-1} g_j \varepsilon_j^{3/2} \Delta\varepsilon_j \le 2 \sum_{j=1}^{N-1} g_j \varepsilon_{j+1} c_j \le 4 \sum_{j=1}^{N-1} f_{j+1}\varepsilon_{j+1} c_{j+1} \le 4\rho E.$$

Note that the inequality

$$\varepsilon_j^{3/2} \Delta\varepsilon_j \le 2\varepsilon_{j+1} c_j, \tag{3.12}$$

has been used which is equivalent to

$$\frac{3\varepsilon_j^{3/2}(\varepsilon_{j+1} - \varepsilon_j)}{\varepsilon_{j+1}\left(\varepsilon_{j+1}^{3/2} - \varepsilon_j^{3/2}\right)} \le 3 \sup_{x\in[0,1[} \frac{x^{3/2}(1-x)}{1-x^{3/2}} = 2.$$

Then, using (3.10), we have

$$\left|\frac{p_i - p_{i-1}}{c_i}\right| \le C \ln(K) f_i. \tag{3.13}$$

Thus, we have for any $i = 1 \cdots N-1$

$$\frac{\partial(f_i/f_{i+1})}{\partial t} = \frac{1}{f_{i+1}}\frac{\partial f_i}{\partial t} - \frac{f_i}{f_{i+1}^2}\frac{\partial f_{i+1}}{\partial t}.$$

Finally, using (3.13), we have

$$\left|\frac{\partial K}{\partial t}\right| \leq CK\ln(K), \tag{3.14}$$

which implies $K(t) \leq K(0)\exp(\exp(Ct))$ and this ends the proof. ■

Remark 3.2. Taking an arithmetic mean for g_i terms, that is $g_i = (f_i + f_{i+1})/2$ (like in the work of Berezin *et al.*; see [2]) leads to the function K (see [4]) for an estimate of the form

$$\left|\frac{\partial K}{\partial t}\right| \leq CK^2\ln(K). \tag{3.15}$$

Since this differential equation has no global solution in time, it cannot be proved that the semi-discretization described in [2] has a global positive solution.

Remark 3.3. An alternative proof can be given following the ideas presented in next section (see Proposition 4.1). Indeed, we show that the discrete collision term can always be written as

$$\mathrm{FP}_i^s = G_i + K_i f_i,$$

where G_i is positive (gain term) and K_i is bounded by some constant C. So that for all i we have

$$\frac{df_i}{dt} \geq -Cf_i.$$

Such inequality implies that the weights f_i cannot vanish in finite time.

4. THE TIME-DISCRETIZED FPLE

In this section, the bars denote the various quantities (like f_i) at time $t_{n+1} = t_n + \Delta t$ defined recursively. Let us introduce the following time explicit scheme

$$\overline{f_i} = f_i + \Delta t\mathrm{FP}_i^s, \tag{4.1}$$

where FP_i^s is defined by (3.7), of the form $p_i - p_{i-1}/c_i$ for $i = 2, \ldots, N-1$, and p_i can be written in the form

$$p_i = g_i(\mathrm{D}(\ln f)_i A_i - B_i) \quad \forall i = 1\cdots N-1,$$

with

$$A_i = \sum_{j=1}^{N-1} g_j k_{i,j}\Delta\varepsilon_j \quad \text{and} \quad B_i = \sum_{j=1}^{N-1} g_j \mathrm{D}(\ln f)_j k_{i,j}\Delta\varepsilon_j. \tag{4.2}$$

4.1. *Cost and Implementation of the Algorithm*

The particular form of the discrete function $k_{i,j}$,

$$k_{i,j} = \begin{cases} \varepsilon_{i+1/2}^{3/2} & \forall i < j, \\ \varepsilon_{j+1/2}^{3/2} & \forall j \geq i, \end{cases}$$

permits us to evaluate all the N terms A_i and B_i in $O(N)$ operations. Indeed, we have, using the definition of $k_{i,j}$,

$$A_i = \varepsilon_{i+1/2}^{3/2} \sum_{N-1 \geq j > i} g_j \Delta\varepsilon_j + \sum_{1 \leq j \leq i} g_j \varepsilon_{j+1/2}^{3/2} \Delta\varepsilon_j \tag{4.3}$$

and

$$B_i = \varepsilon_{i+1/2}^{3/2} \sum_{N-1 \geq j > i} g_j \mathrm{D}(\ln f) \Delta\varepsilon_j + \sum_{1 \leq j \leq i} g_j \mathrm{D}(\ln f) \varepsilon_{j+1/2}^{3/2} \Delta\varepsilon_j. \tag{4.4}$$

Obviously, A_i and B_i can be evaluated using three loops. The detailed algorithm for the computation of all the terms p_i reads :

ALGORITHM 4.1.

$\alpha_1 = g_1 * \varepsilon_{1+1/2}^{3/2} * \Delta\varepsilon_1$;
$\gamma_1 := g_1 * \mathrm{D}(\ln f)_1 * \varepsilon_{1+1/2}^{3/2} * \Delta\varepsilon_1$;
for $i := 2$ to $N-1$ **do**
  $\alpha_i := \alpha_{i-1} + g_i * \varepsilon_{i+1/2}^{3/2} * \Delta\varepsilon_i$;
  $\gamma_i := \gamma_{i-1} + g_i * \mathrm{D}(\ln f)_i * \varepsilon_{i+1/2}^{3/2} * \Delta\varepsilon_i *$;
end for

$\beta_{N-1} := g_{N-1} * \Delta\varepsilon_{N-1}$;
$\delta_{N-1} := g_{N-1} * \mathrm{D}(\ln f)_{N-1} * \Delta\varepsilon_{N-1}$;
for $i := N-2$ to 1 **do**
  $\beta_i := \beta_{i+1} + g_i * \Delta\varepsilon_i$;
  $\delta_i := \delta_{i+1} + g_i * \mathrm{D}(\ln f)_i * \Delta\varepsilon_i$;
end for

for $i := 1$ to $N-1$ **do**
  $A_i := \varepsilon_{i+1/2}^{3/2} * \beta_i + \alpha_i$;
  $B_i := \varepsilon_{i+1/2}^{3/2} * \delta_i + \gamma_i$;
  $p_i := g_i * ((\mathrm{D} \ln f)_i * A_i + B_i)$;
end for

4.2. *Time Step Restriction for Positivity and Entropy Decay*

The main questions about the time explicit scheme (4.1) concern positivity and entropy decay property. By positivity of the scheme, we mean that the terms $\overline{f_i}$ are positive if the terms f_i are positive and by entropy decay, the property $H(\overline{f_i}) \leq H(f_i)$, where the discretized entropy H is defined by (3.8).

We obtain a time step limitation in order that the scheme remains positive and the entropy decays. The last question is related to the series of time steps; its divergence provides a positive and entropy decaying time-discretized solution for any arbitrary large time. We perform the analysis for two natural grids which are used for the numerical examples presented later.

The first one is a **uniform grid in velocity**, where the nodes of the grid are defined by the sequence $\varepsilon_i = (i-1)^2\Delta v^2$ with $\Delta v = \sqrt{\varepsilon_0}/(N-1)$, N being the number of grid points. For this choice, the geometric quantities used in the definition of the scheme are

- $\Delta\varepsilon_i = (\Delta v)^2(2i-1)$.
- $c_i = (3(i-1)^2+1)\Delta v^3/3$ except for $i = N$, where $c_N = (3N^2-9N+7)\Delta v^3/3$.
- $\varepsilon_{i+1/2} = (2i^2-2i+1)\Delta v^2/2$.

The second type is a **uniform grid in energy**, i.e. $\varepsilon_i = (i-1)\Delta\varepsilon$ with $\Delta\varepsilon = \varepsilon_0/(N-1)$ and the geometric quantities used reads now:

- $\Delta\varepsilon_i = \Delta\varepsilon$.
- $c_i = (i^{3/2}-(i-2)^{3/2})\Delta\varepsilon^{3/2}/3$ except for $i = 1$ and for $i = N$ for which $c_1 = \Delta\varepsilon^{3/2}$ and $c_N = ((N-1)^{3/2}-(N-2)^{3/2})\Delta\varepsilon^{3/2}$ respectively. It is easy to check that we have the following lower bound for c_i which will be usefull later: $c_i \geq \Delta\varepsilon^{3/2}((i-1)+\sqrt{i(i-1)})/(3\sqrt{N})$ for $i = 2\cdots N-1$, and for $i = N$, $c_N \geq \Delta\varepsilon^{3/2}(N-3/2+\sqrt{(N-1)(N-2)})/(3\sqrt{N})$.
- $\varepsilon_{i+1/2} = (2i-1)\Delta\varepsilon/2$.

For these two grids, we obtain sufficient conditions for the time step in order to ensure positivity and entropy decay. We summarize this result as

PROPOSITION 4.1. *For each grid considered above, there exists a constant C which depends only on the density ρ, the entropy H, and the length ε_0, such that the scheme* (4.1) *is positive and entropy decaying under a time step restriction of the form $\Delta t \leq C\Delta v^2$ for the* **uniform grid in velocity** *or $\Delta t \leq C\Delta\varepsilon^2$ for the* **uniform grid in energy**.

Proof. Let us first exhibit a sufficient condition on the time step to guarantee entropy decay. Suppose that there exists a time step Δt^0 such that for all $\Delta t \in [0, \Delta t^0[$ all the terms $\bar{f}_j$ are positive. Then, using the definition (3.8), the entropy associated with the scheme (4.1) is

$$\bar{H} = H(\Delta t) = \sum_{j=1}^{N} c_j\,\bar{f}_j \ln(\bar{f}_j) \geq \sum_{j=1}^{N} c_j\big(f_j + \Delta t \mathrm{FP}_j^s\big)\ln\big(f_j + \Delta t \mathrm{FP}_j^s\big). \tag{4.5}$$

Then, we have, using the inequality $\ln(1+x) < x\ \forall x > -1$ and the conservation of the mass,

$$H(\Delta t) \leq H(0) + \Delta t\sum_{i=1}^{N} c_i \mathrm{FP}_i^s \ln(f_i) + \Delta t^2\sum_{i=1}^{N} c_i\big(\mathrm{FP}_i^s\big)^2/f_i \stackrel{\text{def}}{=} \tilde{H}(\Delta t)$$

for all $\Delta t \in [0, \Delta t^0[$. Thus, a sufficient condition for the entropy decay is to choose Δt such that $\tilde{H}(\Delta t) \leq \tilde{H}(0) = H(0)$, or equivalently,

$$\Delta t \leq \frac{-\sum_{i=1}^{N} c_i \mathrm{FP}_i^s \ln(f_i)}{\sum_{i=1}^{N} c_i\big(\mathrm{FP}_i^s\big)^2/f_i}.$$

By construction, we have

$$-\sum_{i=1}^{N} c_i \,\mathrm{FP}_i^s \ln(f_i) = \frac{1}{2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} g_i g_j k_{i,j}\,\Delta\varepsilon_i\,\Delta\varepsilon_j\, D_{i,j}^2 \geq 0.$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^{N} c_i \left(\mathrm{FP}_i^s\right)^2/f_i &\leq \frac{p_1^2}{c_1 f_1} + \frac{p_{N-1}^2}{f_N c_{N-1}} + \sum_{i=2}^{N-1}\frac{1}{c_i f_i}(p_i - p_{i-1})^2\\ &\leq \frac{p_1^2}{c_1 f_1} + \frac{p_{N-1}^2}{f_N c_{N-1}} + \sum_{i=2}^{N-1}\frac{2}{c_i f_i}\left(p_i^2 + p_{i-1}^2\right).\end{aligned}$$

Using the definition of the term p_i, we have

$$p_i^2 = \left(\sum_{i=1}^{N-1} g_i g_j k_{i,j}\,\Delta\varepsilon_j\, D_{i,j}\right)^2 \leq \left(\sum_{i=1}^{N-1} g_i g_j k_{i,j}\,\Delta\varepsilon_j\right)\left(\sum_{i=1}^{N-1} g_i g_j k_{i,j}\,\Delta\varepsilon_j\, D_{i,j}^2\right),$$

and using that $g_i \leq 2 f_i$, we obtain

$$\begin{aligned}\frac{p_i^2}{f_i c_i} &\leq \frac{2}{c_i\,\Delta\varepsilon_i}\left(\sum_{i=1}^{N-1} g_j k_{i,j}\,\Delta\varepsilon_j\right)\left(\sum_{i=1}^{N-1} g_i g_j k_{i,j}\,\Delta\varepsilon_j\,\Delta\varepsilon_i\, D_{i,j}^2\right)\\ &\leq \sup_{i=1\cdots N}\left(\frac{2}{c_i\,\Delta\varepsilon_i}\sum_{i=1}^{N-1} k_{i,j} g_j\,\Delta\varepsilon_j\right)\left(\sum_{i=1}^{N-1} g_i g_j k_{i,j}\,\Delta\varepsilon_j\,\Delta\varepsilon_i\, D_{i,j}^2\right).\end{aligned}$$

The same estimate can be obtained for $p_{i-1}/f_i c_i$, since the sequence $\Delta\varepsilon_i$ is increasing. By summing these inequalities, one obtains

$$\sum_{i=1}^{N} c_i \left(\mathrm{FP}_i^s\right)^2/f_i \leq \sup_{i=1\cdots N}\left(\frac{16}{c_i\,\Delta\varepsilon_i}\sum_{j=1}^{N-1} k_{i,j} g_j\,\Delta\varepsilon_j\right)\left(-\sum_{i=1}^{N} c_i\,\mathrm{FP}_i^s \ln(f_i)\right).$$

Finally, the time step has to satisfy

$$\Delta t.\sup_{i=1}^{N}\left(\frac{16}{c_i\,\Delta\varepsilon_i}\sum_{j=1}^{N-1} k_{i,j} g_j\,\Delta\varepsilon_j\right) \leq 1. \tag{4.6}$$

Equation (4.6) gives the time step limitation used for the numerical examples. We must now find an upper bound of the denominator of (4.6). We detail the majoration for the uniform grid in velocity, since for the uniform grid in energy it follows the same lines. The problem is to estimate the above terms independently of i. For the uniform grid in the velocity variable, using the expressions of $\Delta\varepsilon_i$ and c_i and since $\varepsilon_{i+1/2}^{3/2} \geq k_{i,j}$, we have for the denominator of (4.6)

$$A_i = \left(\sum_{j=1}^{N-1} k_{i,j} g_j\,\Delta\varepsilon_j\right) \leq \varepsilon_{i+1/2}^{3/2}\left(\sum_{j=1}^{N-1} g_j\,\Delta\varepsilon_j\right). \tag{4.7}$$

Using the definitions of c_i, $\Delta\varepsilon_i$, and $\varepsilon_{i+1/2}$, one also has

$$\frac{\varepsilon_{i+1/2}^{3/2}}{c_i\,\Delta\varepsilon_i} = \frac{3}{2\sqrt{2}}\left(\frac{(2i^2-2i+1)^{3/2}}{(3(i-1)^2+1)(2i-1)}\right)\frac{1}{\Delta v^2}.$$

Since the term depending on i is bounded, we have a time step of the form

$$\Delta t \le C\left(\sum_{j=1}^{N-1} g_j\,\Delta\varepsilon_j\right)(\Delta v)^2,$$

where C is a constant independent of the data. Let us now bound the term $\sum_{j=1}^{N-1} g_j\,\Delta\varepsilon_j$. By the Cauchy–Schwarz inequality, we have

$$\sum_{j=1}^{N-1} g_j\,\Delta\varepsilon_j \le \sqrt{\sum_{j=1}^{N-1} g_j^2 c_j}\sqrt{\sum_{j=1}^{N-1}\left(\Delta\varepsilon_j^2/c_j\right)}. \tag{4.8}$$

By replacing $\Delta\varepsilon_j$ and c_j by their values, it is easy to check that $\sqrt{\sum_{j=1}^{N-1}(\Delta\varepsilon_j^2/c_j)}$ is bounded by a constant which depends only on the length of the domain ε_0. On the other hand, one defines the discrete L_2 norm

$$\frac{1}{2}\sqrt{\sum_{j=1}^{N-1} g_j^2 c_j} \le \sqrt{\sum_{j=1}^{N} f_j^2 c_j} \stackrel{\text{def}}{=} \|f\|_2.$$

Finally, the scheme is entropy decaying under a condition for the time step of the form

$$\Delta t \le C(\varepsilon_0, \|f\|_2)\Delta v^2.$$

For the uniform grid in energy, as indicated above, the majoration can be carried out using the same techniques. This gives the inequalities

$$A_i = \left(\sum_{j=1}^{N-1} k_{i,j}\, g_j\,\Delta\varepsilon_j\right) \le \varepsilon_{i+1/2}\left(\sum_{j=1}^{N-1}\varepsilon_{i+1/2}^{1/2} g_j\,\Delta\varepsilon_j\right), \tag{4.9}$$

instead of (4.7). It is also necessary to use the lower bound on c_i. We obtain the same kind of time step restriction for entropy decaying as for the uniform grid in velocity, but with $\Delta\varepsilon^2$ instead of Δv^2.

Let us now exhibit a sufficient condition on the time step to guarantee the positivity of the scheme (4.1). Using the notation defined above, we can write the terms FP_i^s as a sum of a positive term G_i and a pseudo-loss term (which is not necessarily negative) of the form $K_i f_i$, with bounded coefficients K_i. Indeed, FP_i^s reads

$$\frac{1}{c_i}\left(\frac{A_i g_i}{\Delta\varepsilon_i}\ln\left(\frac{f_{i+1}}{f_i}\right) + \frac{A_{i-1} g_{i-1}}{\Delta\varepsilon_{i-1}}\ln\left(\frac{f_{i-1}}{f_i}\right) - g_i B_i - g_{i-1} B_{i-1}\right),$$

where A_i and B_i are defined by (4.3) and (4.2), respectively. First, it is easy to check that all the terms B_i are bounded; then, using (3.10), $B_i g_i/(f_i c_i)$ are bounded and are taking into account in K_i. The same result holds for $B_{i-1} g_{i-1}/(f_{i-1} c_{i-1})$.

Consider now the term containing A_i. If $f_{i+1} \geq f_i$, this term is positive, then it is taken into account in the gain term G_i. On the contrary, since $A_i \geq 0$ and $g_i \geq 0$, this term is negative and in such case, we have

$$\left| g_i \ln\left(\frac{f_{i+1}}{f_i}\right) \right| \leq 2 \sup_{1 \geq x \geq 0} |x \ln(x)| \; f_i = 2e^{-1} f_i \leq 2 f_i,$$
$$\left| \frac{1}{c_i \, f_i \, \Delta\varepsilon_i} A_i g_i \ln\left(\frac{f_{i+1}}{f_i}\right) \right| \leq \frac{2 A_i}{c_i \, \Delta\varepsilon_i}.$$

This term taken into account in K_i. It is straightforward to show the same result for

$$\frac{1}{c_i \, \Delta\varepsilon_{i-1}} A_{i-1} g_{i-1} \ln\left(\frac{f_{i-1}}{f_i}\right).$$

Therefore, we have

$$\mathrm{FP}_i^s = G_i + K_i \, f_i$$

with $G_i \geq 0$ and K_i bounded. Then, $\overline{f_i} = f_i + \mathrm{FP}_i^s$ is positive provided that

$$\Delta t \leq \left(\max_i |K_i| \right)^{-1},$$

and it is easy to check that

$$\max_i |K_i| \leq 2 \left(\max_i \left| \frac{A_i}{c_i \, \Delta\varepsilon_i} \right| + \max_i \left| \frac{B_i}{c_i} \right| \right).$$

Then, under the condition

$$\Delta t \leq \frac{1}{2} \left(\max_i \left| \frac{A_i}{c_i \, \Delta\varepsilon_i} \right| + \max_i \left| \frac{B_i}{c_i} \right| \right)^{-1}, \tag{4.10}$$

the scheme is positive. Let us now detail for the uniform grid in v such a time restriction. Recall the inequality obtained from (4.7)

$$\frac{A_i}{c_i \, \Delta\varepsilon_i} \leq C(\varepsilon_0, \| f \|_2)/(\Delta v)^2. \tag{4.11}$$

For the terms $|B_i/c_i|$ we use

$$\left| \frac{B_i}{c_i} \right| \frac{\varepsilon_{i+1/2}}{c_i} \sum_{j=1}^{N-1} \varepsilon_{j+1/2}^{1/2} \, g_j |(D \ln f)_j| \Delta\varepsilon_j \leq \frac{C}{\Delta v} \sum_{j=1}^{N-1} \varepsilon_{j+1/2}^{1/2} \, g_j |(D \ln f)_j| \Delta\varepsilon_j. \tag{4.12}$$

Using (3.10) and the fact that the sequence $\Delta\varepsilon_j$ is increasing, we have

$$\begin{aligned}
\sum_{j=1}^{N-1} \varepsilon_{j+1/2}^{1/2} \, g_j |(D \ln f)_j| \Delta\varepsilon_j &\leq \sum_{j=1}^{N-1} \frac{\varepsilon_{j+1/2}^{1/2}}{\Delta\varepsilon_j} g_j (|\ln f_j| + |\ln f_{j+1}|) \Delta\varepsilon_j \\
&\leq 4 \sum_{j=1}^{N-1} \frac{\varepsilon_{j+1/2}^{1/2}}{\Delta\varepsilon_j} f_j |\ln f_j| \Delta\varepsilon_j + 2 f_N |\ln(f_N)| \varepsilon_{N-1/2}^{1/2} \\
&\leq \frac{C'}{\Delta v} \sum_{j=1}^{N} f_j |\ln f_j| \Delta\varepsilon_j.
\end{aligned}$$

Then, using the Cauchy–Schwarz inequality as for (4.8), we obtain

$$\sum_{j=1}^{N-1} \Delta\varepsilon_j\, f_j\, |\ln f)_j| \leq \sqrt{\sum_{j=1}^{N-1} f_j^2\, |\ln f_j|^2 c_j} \sqrt{\sum_{j=1}^{N-1} \frac{\Delta\varepsilon_j^2}{c_j}} \leq C(\varepsilon_0, \| f \ln(f)\|_2). \quad (4.13)$$

Collecting all the results, we show that, in the case of an uniform grid in v, there exists a constant C such that for any time step satisfying

$$\Delta t \leq C(\varepsilon_0, \| f \ln(f)\|_2, \| f \|_2)\Delta v^2 \quad (4.14)$$

the scheme is positive For the case of an uniform grid in ε, we do not detail the calculations. One uses an estimate of the form (4.7) instead of (4.11) to obtain an upper bound of $A_i/(c_i\Delta\varepsilon_i)$. For $B_i/(c_i)$, one proceeds exactly as for the uniform grid in velocity. Finally, we obtain (4.14) with $\Delta\varepsilon^2$ instead of Δv^2.

For each type of grid, by taking $C = \min(C_1, C_2)$, we obtain the desired result. ■

Remark 4.2. On the numerical examples, $\max_i(f_i)$ (and consequently the L_2 norm $\| f \|_2$) appears to be bounded not only uniformly in time (for a fixed N), which can be proved using the mass conservation, but also independently of the mesh size. This remains to be proved in order to approach the problem of the convergence.

Remark 4.3. As we will see in the next section on a numerical example, preserving the positivity only, by taking a time step of the form $\Delta t = \alpha\Delta t^0$, where Δt^0 is the maximum allowable time step satisfying

$$f_i + \Delta t^0 \mathrm{FP}_i^s \geq 0 \quad \forall i \in I$$

with the CFL factor α equal to 0.5, for example, does not permit to avoid oscillations. However, (4.6) and (4.10) yield to a nonoscillatory scheme even if there is no maximum principle for the nonlinear FPLE.

5. NUMERICAL TEST FOR THE FPLE

The numerical test presented now is extracted from the work of Rosenbluth *et al.* [19] and has been used by Larroche *et al.* [12] and Frenod and Lucquin [11] to test numerical methods for the Fokker–Planck–Landau equation. The initial data is given by

$$f^0(\varepsilon) = 0.01 \exp(-10[(\sqrt{\varepsilon} - 0.3)/0.3]^2). \quad (5.1)$$

We will show the entropy, the Linnick functionnal and the distribution function at time $t \in \{9, 36, 81, 144, 225, 324, 441, 576, 729, 900\}$. The Linnick functionnal is defined by

$$L(t) = \int_{v\in\mathbb{R}^3} \frac{(\nabla_v f)^2}{f}\, dv \quad (5.2)$$

and for an isotropic function this reduces to

$$L(t) = \int_{\varepsilon\geq 0} \left(\frac{\partial f}{\partial\varepsilon}\right)^2 \frac{\varepsilon^{3/2}}{f}\, d\varepsilon. \quad (5.3)$$

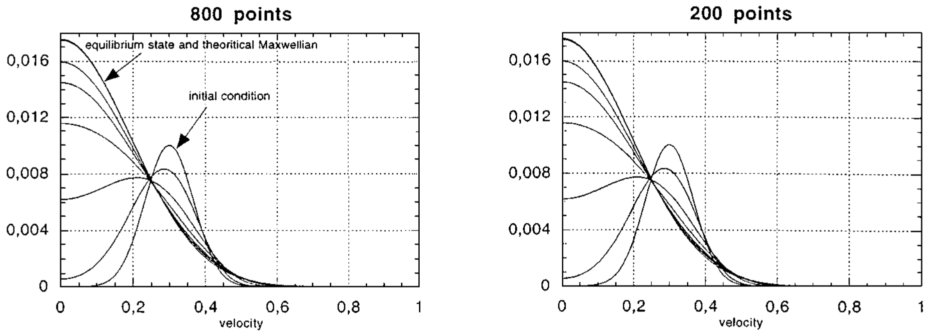

FIG. 1. Distribution function for the uniform grid in velocity.

The Linnick functionnal is known to be decreasing in time for the Boltzmann equation and the linear Fokker–Planck equation [3, 20, 9, 10]. Since the nonlinear FPLE is the so-called grazing collisions limit of the Boltzmann equation, one can expect that it is also decreasing for FPLE. For initial data (5.1) the Linnick functionnal is actually time-decreasing. Moreover, this functionnal illustrates very well the instabilities due to a nonentropy decaying scheme.

The tests run with two types of meshes, a uniform grid in the modulus of the velocity $v=\sqrt{\varepsilon}$ and the other in the energy variable ε already described in the preceeding section. For the two type of grids we take either 200 or 800 points of discretization and $\varepsilon_0 = 1$. The computations were carried out with a global time step equal to 1 and using subcycling inside each time step in order to preserve the positivity of the solution and to respect the entropy condition (4.6) with a CFL factor equal to 4.

The tests have been performed on a personnal Apple computer, with a 160 Mhz PPC 603 chip. The cost of evaluating the solution during a time step for the uniform grid in energy (resp. in velocity) is about 0.09 s (resp. 0.05 s) for 200 cells and 7.46 s (resp. 3.71 s) for 800 cells; 900 time steps are performed. Note that the increase of the computationnal time is in good agreement with the theoretical estimate, since it is around a cubic function of the number N of points ($\Delta t \leq C/N^2$ and linear cost $O(N)$ of the algorithm).

Figures 1 and 2 show the time relaxation of the distribution function at various times: initial condition, time $t \in \{9, 36, 81, 144, 225, 900\}$, and the equilibrium state. Figures 3 and 4 show for this thermalization experiment the time relaxation of the H and Linnick

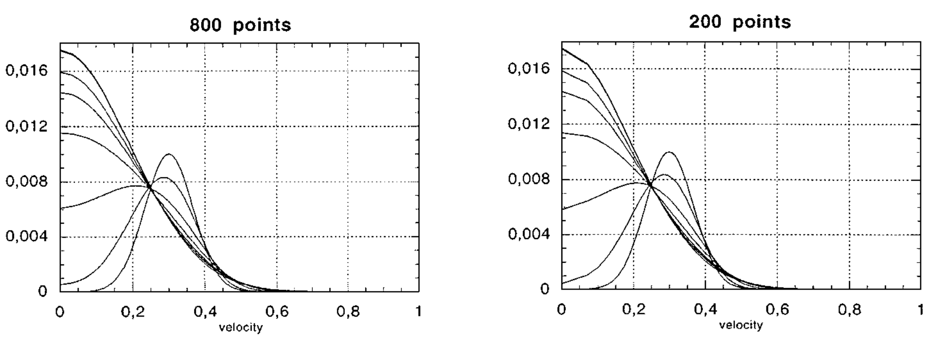

FIG. 2. Distribution function for the uniform grid in energy.

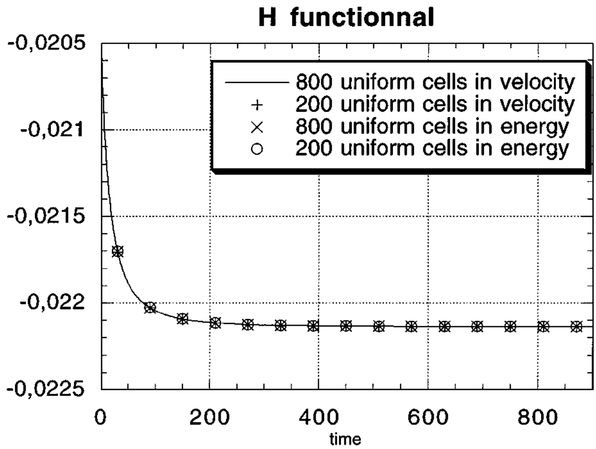

FIG. 3. Entropy.

functionnals. One can observe for these quantities that the results are very close to each other.

We also run a simulation using 200 cells uniformly distributed in energy and by only imposing the positivity of the solution using at each iteration half of the maximal allowable time step that guarantees the positivity. The run is only three times faster than the ones made with a CFL equal to 4 for the entropy condition 4.6.

Figures 5, 6, and 7 show the relaxation of the H and Linnick functionnals and the distribution function at $t = 81$. For the entropy, Fig. 5 compares the result with the entropy obtained for the same grid and the entropy-decaying scheme. The noisy curve corresponds to the “nonentropy-decaying” computation. For the first time steps, the two curves are very close. For large time, it is clear that the “nonentropy-decaying” computation has some difficulties reaching the equilibrium state, but the result does not seem too bad. On the Fig. 7 we plot the distribution function obtained with the two schemes at $t = 81$. The results are qualitatively the same for other times except for the small ones. The domain of oscillations

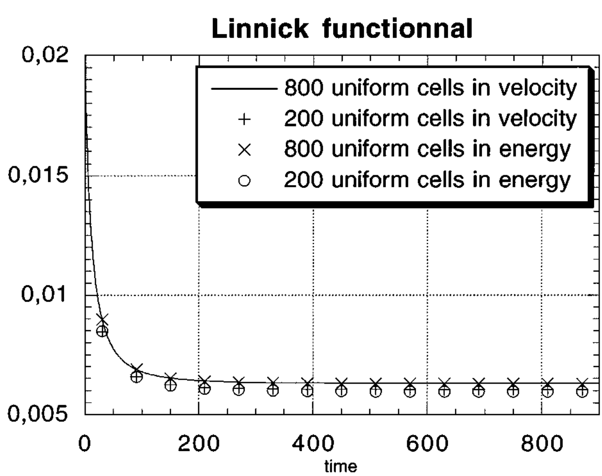

FIG. 4. Linnick functionnal.

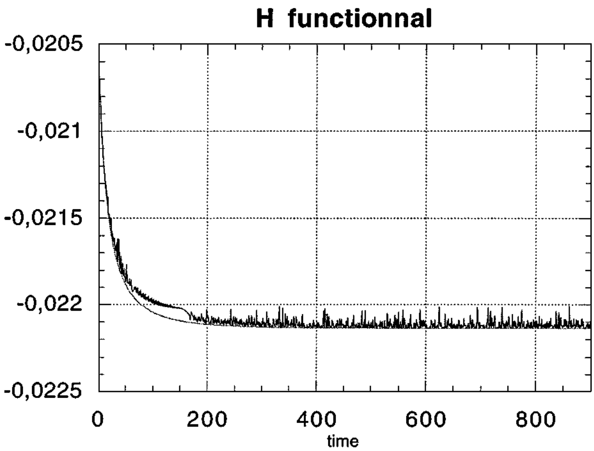

FIG. 5. Entropy: Comparison between entropy and nonentropy computations.

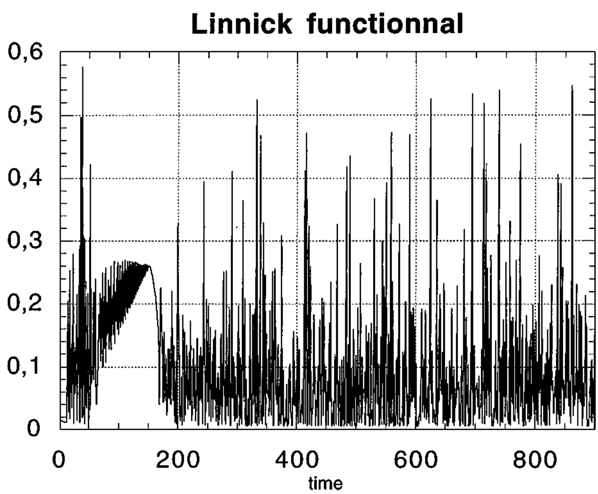

FIG. 6. Nonentropy computation: Linnick functionnal.

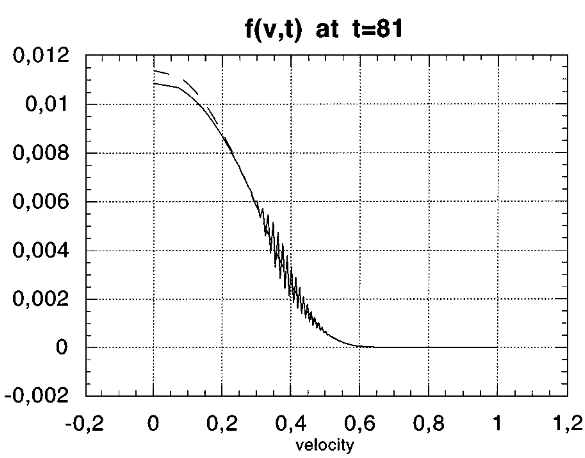

FIG. 7. Comparison between entropy and nonentropy computation for the distribution function at $t = 81$.

is independent of time and of the number of grid points. At large time, these oscillations persist but are damped. The main difference can be seen on the Linnick functional for which the results appear totally randomized, with no relationship to the ”exact” (entropy decaying one) behaviour. Note that the same type of computation with a uniform grid in velocity produces a totally different behaviour. The distribution function is uniquely noisy near $v = 0$ which explains, by recalling that the measure of integration is $\sqrt{\varepsilon}\, d\varepsilon$, that functionnals of the distribution function have a correct time relaxation.

6. CONCLUSIONS

We provide for the simplest case of the isotropic, homogeneous, and Coulomb Fokker–Planck–Landau equation, a complete analysis of a conservative and entropy decaying numerical scheme. This scheme is very close to the scheme proposed in [2] since the modification consists in taking the harmonic average, instead of the arithmetic one for the evaluation of g_i. The main advantage of this scheme is to provide rigorously for the first time the existence of solutions for the semi-discretized model and time step restrictions to ensure positivity and entropy decaying of the scheme. We show that relaxing these time step conditions provides suspicious numerical results of FPLE for any time (see plots of Linnick functional).

We refer to [4] for similar analysis for the linear and nonlinear FPLE given by (2.1).

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